Relativistic Dynamics

Michael Fowler

UVa Physics

Physics 252 Home Page
Link to Previous Lecture

The Story So Far: A Brief Review

The first coherent statement of what physicists now call *relativity* was Galileo's observation almost four hundred years ago that if you were in a large closed room, you could not tell by observing how things move-living things, thrown things, dripping liquids-whether the room was at rest in a building, say, or below decks in a large ship moving with a steady velocity. More technically (but really saying the same thing!) we would put it that *the laws of motion are the same in any inertial frame*. That is, these laws really only describe *relative* positions and velocities. In particular, they do not single out a special inertial frame as the one that's "really at rest". This was later all written down more formally, in terms of *Galilean transformations*. Using these simple linear equations, motion analyzed in terms of positions and velocities in one inertial frame could be translated into any other. When, after Galileo, Newton wrote down his Three Laws of Motion, they were of course invariant under the Galilean transformations, and valid in any inertial frame.
About two hundred years ago, it became clear that light was not just a stream of particles (as Newton had thought) but manifested definite *wavelike* properties. This led naturally to the question of what, exactly, was waving, and the consensus was that space was filled with an *aether*, and light waves were ripples in this all-pervading aether analogous to sound waves in air. Maxwell's discovery that the equations describing electromagnetic phenomena had wavelike solutions, and predicted a speed which coincided with the measured speed of light, suggested that electric and magnetic fields were stresses or strains in the aether, and Maxwell's equations were presumably only precisely correct in the frame in which the aether was at rest. However, very precise experiments which should have been able to detect this aether all failed.
Almost one hundred years ago, Einstein suggested that maybe *all* the laws of physics were the same in *all* inertial frames, generalizing Galileo's pronouncements concerning motion to include the more recently discovered laws of electricity and magnetism. This would imply there could be no special "really at rest" frame, even for light propagation, and hence no aether. This is a very appealing and very simple concept: the same laws apply in all frames. What could be more reasonable? As we have seen, though, it turns out to clash with some beliefs about space and time deeply held by everybody encountering this for the first time. The central prediction is that since the speed of light follows from the laws of physics (Maxwell's equations) and some simple electrostatic and magnetostatic experiments, which are clearly frame-independent, *the speed of light is the same in all inertial frames*. That is to say, the speed of a *particular flash* of light will always be measured to be $3\ 10^8$ meters per second even if measured by different observers moving rapidly relative to each other, where each observer measures the speed of the flash relative to himself. Nevertheless, experiments have show again and again that Einstein's elegant insight is right, and everybody's deeply held beliefs are wrong.
We have discussed in detail the *kinematical* consequences of Einstein's postulate-how measurements of position, time and velocity in one frame relate to those in another, and how apparent paradoxes can be resolved by careful analysis. So far, though, we have not thought much about dynamics. We know that Newton's Laws of Motion were invariant under the Galilean transformations between inertial frames. We now know that the Galilean transformations are in fact *incorrect* except in the low speed nonrelativistic limit. Therefore, we had better look carefully at Newton's Laws of Motion in light of our new knowledge.

Newton's Laws Revisited

Newton's First Law, the Principle of Inertia, that an object subject to no external forces will continue to move in a straight line at steady speed, is equally valid in special relativity. Indeed, it is the defining property of an inertial frame that this *is* true, and the content of special relativity is transformations between such frames.

Newton's Second Law, stated in the form force = mass x acceleration, cannot be true as it stands in special relativity. This is evident from the formula we derived for addition of velocities. Think of a rocket having many stages, each sufficient to boost the remainder of the rocket (including the unused stages) to *c/2* from rest. We could fire them one after the other in a carefully timed way to generate a continuous large force on the rocket, which would get it to *c/2* in the first firing. If the acceleration continued, the rocket would very soon be exceeding the speed of light. Yet we know from the addition of velocities formula that in fact the rocket never reaches *c*. Evidently, Newton's Second Law needs updating.

Newton's Third Law, action = reaction, also has problems. Consider some attractive force between two rapidly moving bodies. As their distance apart varies, so does the force of attraction. We might be tempted to say that the force of *A* on *B* is the opposite of the force of *B* on *A*, at each instant of time, but that implies *simultaneous* measurements at two bodies some distance from each other, and if it happens to be true in *A*'s inertial frame, it won't be in *B*'s.

Conservation Laws

In nonrelativistic Newtonian physics, the Third Law tells us that two interacting bodies feel equal but opposite forces from the interaction. Therefore from the Second Law, the rate of change of momentum of one of the bodies is equal and opposite to that of the other body, thus the *total* rate of change of momentum of the system caused by the interaction is zero. Consequently, for any *closed* dynamical system (no outside forces acting) *the total momentum never changes*. This is the *law of conservation of momentum*. It does *not* depend on the details of the forces of interaction between the bodies, only that they be equal and opposite.

The other major dynamical conservation law is the conservation of energy. This was not fully formulated until long after Newton, when it became clear that frictional heat generation, for example, could quantitatively account for the apparent loss of kinetic plus potential energy in actual dynamical systems.

Although these conservation laws were originally formulated within a Newtonian worldview, their very general nature suggested to Einstein that they might have a wider validity. Therefore, as a working hypothesis, he assumed them to be satisfied *in all inertial frames*, and explored the consequences. We follow that approach.

Momentum Conservation on the Pool Table

As a warm-up exercise, let us consider conservation of momentum for a collision of two balls on a pool table. We draw a chalk line down the middle of the pool table, and shoot the balls close to, but on opposite sides of, the chalk line from either end, at the same speed, so they will hit in the middle with a glancing blow, which will turn their velocities through a small angle. In other words, if initially we say their (equal magnitude, opposite direction) velocities were parallel to the *x*-direction -- the chalk line -- then after the collision they will also have equal and opposite small velocities in the *y*-direction. (The *x*-direction velocities will have decreased very slightly).

A Symmetrical Spaceship Collision

Now let us repeat the exercise on a grand scale. Suppose somewhere in space, far from any gravitational fields, we set out a string one million miles long. (It could be between our two clocks in the time dilation experiment). This string corresponds to the chalk line on the pool table. Suppose now we have two identical spaceships approaching each other with equal and opposite velocities parallel to the string from the two ends of the string, aimed so that they suffer a slight glancing collision when they meet in the middle. It is evident from the symmetry of the situation that momentum is conserved in both directions. In particular, the rate at which one spaceship moves away from the string after the collision - its *y*-velocity - is equal and opposite to the rate at which the other one moves away from the string.

But now consider this collision as observed by someone in one of the spaceships, call it *A*. (Remember, momentum must be conserved in *all* inertial frames -- they are all equivalent -- there is nothing special about the frame in which the string is at rest.) Before the collision, he sees the string moving very fast by the window, say a few meters away. After the collision, he sees the string to be moving away, at, say, 15 meters per second. This is because spaceship *A* has picked up a velocity perpendicular to the string of 15 meters per second. Meanwhile, since this is a completely symmetrical situation, an observer on spaceship *B* would certainly deduce that her spaceship was moving away from the string at 15 meters per second as well.

Just how symmetrical is it?

The crucial question is: *how fast does an observer in spaceship A see spaceship B to be moving away from the string?* Let us suppose that relative to spaceship *A*, spaceship *B* is moving away (in the *x*-direction) at 0.6*c*. First, recall that distances perpendicular to the direction of motion are not Lorentz contracted. Therefore, when the observer in spaceship *B* says she has moved 15 meters further away from the string in a one second interval, the observer watching this movement from spaceship *A* will agree on the 15 meters - but disagree on the one second! He will say her clocks run slow, so as measured by his clocks 1.25 seconds will have elapsed as she moves 15 meters in the *y*-direction.
It follows that, as a result of time dilation, this collision as viewed from spaceship *A* does *not* cause equal and opposite velocities for the two spaceships in the *y*-direction. Initially, both spaceships were moving parallel to the *x*-axis - there was zero momentum in the *y*-direction. Consider *y*-direction momentum conservation in the inertial frame in which *A* was initially at rest. An observer in that frame measuring *y*-velocities after the collision will find *A* to be moving at 15 meters per second, *B* to be moving at -0.8 x 15 meters per second in the *y*-direction. So how can we argue there is zero total momentum in the *y*-direction *after* the collision, when the identical spaceships do *not* have equal and opposite velocities?

Einstein rescues Momentum Conservation

Einstein was so sure that momentum conservation must always hold that he rescued it with a bold hypothesis: the mass of an object must depend on its speed! In fact, the mass must increase with speed in just such a way as to cancel out the lower *y*-direction velocity resulting from time dilation. That is to say, if an object at rest has a mass *M*, moving at a speed *v* it will have a mass *M*/sqrt(1 - *v*∑/*c*∑). Note that this is an undetectably small effect at ordinary speeds, but as an object approaches the speed of light, the mass increases without limit!
Of course, we have taken a very special case here - a particular kind of collision. The reader might well wonder if the same mass correction would work in other types of collision, for example a straight line collision in which a heavy object rear-ends a lighter object. The algebra is straightforward, if tedious, and it is found that this mass correction factor does indeed ensure momentum conservation for any collision in all inertial frames.

Mass Really *Does* Increase with Speed

Deciding that masses of objects must depend on speed like this seems a heavy price to pay to rescue conservation of momentum! However, it is a prediction that is not difficult to check by experiment. The first confirmation came in 1908, measuring the mass of fast electrons in a vacuum tube. In fact, the electrons in a color TV tube are about half a percent heavier than electrons at rest, and this must be allowed for in calculating the magnetic fields used to guide them to the screen.
Much more dramatically, in modern particle accelerators very powerful electric fields are used to accelerate electrons, protons and other particles. It is found in practice that these particles become heavier and heavier as the speed of light is approached, and hence need greater and greater forces for further acceleration. Consequently, the speed of light is a natural absolute speed limit. Particles are accelerated to speeds where their mass is thousands of times greater than their mass measured at rest, usually called the "rest mass".

Mass and Energy Conservation in Special Relativity

As everyone has heard, in special relativity mass and energy are not separately conserved, in certain situations mass *m* can be converted to energy $E = mc^2$. This equivalence is closely related to the mass increase with speed, as we shall see. In fact, French derives it by analyzing a collision and stipulating mass conservation in the collision, where mass is now the speed-dependent mass discussed above. Part of French's point in doing this is to demonstrate that it follows from very general considerations, for example concepts like force do not need to be introduced. We shall take a less sophisticated route below, in which we find the kinetic energy of an object as it is accelerated in a straight line by a force, such as the electric field acting on an electron in a linear accelerator, and assume Newton's second law is valid provided mass x acceleration is replaced by rate of change of momentum, a more general concept.

Kinetic Energy and Mass for Very Fast Particles

Let's first think about the kinetic energy of a particle traveling close to the speed of light. If a force F acts on the particle accelerating it in its direction of motion, and the particle moves through a distance d in the direction the force is pushing, the force does work Fd, and this must go into the kinetic energy of the particle (assuming as always conservation of energy).
As a warm up, recall the elementary derivation of the kinetic energy $1/2mv\sum$ of an ordinary *non-relativistic* (i.e. slow moving) object of mass m. This can be done by accelerating the mass with a constant force F, and finding the work done by the force (force distance) to get it to speed v from a standing start. The kinetic energy of the mass, $E = 1/2mv\sum$, equals the work done by the force in bringing the mass up to that speed from rest: if it takes time t, and the (uniform) acceleration is a, then the final speed $v = at$, the average speed during acceleration is $1/2v$, so the distance traveled is $1/2vt$, the work done by the force is $1/2Fvt = 1/2mavt = 1/2mv^2$. It is easy to extend this to show that if the mass was initially moving as speed u, the work done by the force is $1/2mv^2 - 1/2mu^2$.
We are now ready to think about the change in kinetic energy of a particle acted on by an accelerating force when the particle is already moving practically at the speed of light, like particles in modern accelerators. The most straightforward way to see what happens is to use Newton's Second Law, which in the form

Force = rate of change of momentum

is still true, but close to the speed of light the speed changes negligibly as the force continues to work-instead, the mass increases. Therefore to an excellent approximation,

Force = (rate of change of mass) c

where as usual c is the speed of light.
To give a concrete example, let us imagine we have a particle moving very close to the speed of light, and we apply an accelerating force F as the particle travels in a straight line through a distance d. (In a particle accelerator, for example, the particle with charge q could enter a region of uniform electric field E and feel a force $F = qE$.)
Then the increase in kinetic energy delta_E is given by

$$\Delta E = Fd = \frac{\Delta m}{\;} cd$$

where we have approximated the rate of change of mass by the actual change over the distance d, assumed small, divided by the time taken to traverse d,

delta_$t = d/c$.

Substituting this value for delta_t in the formula for the increase in kinetic energy delta_E, we find

delta_E = delta_mc^2.

Kinetic Energy and Mass for Slow Particles

Recall that to get momentum to be conserved in all inertial frames, we had to assume an increase of mass

with speed by the factor . This necessarily implies that *even a slow-moving object*

has a tiny mass increase if it is put in motion.
How does this mass increase relate to the kinetic energy? Consider a mass *m*, moving at speed *v*, much less than the speed of light. Its kinetic energy *E* =1⁄2*mv*∑, as discussed above. Its mass is

$$m \Big/ \sqrt{1 - v^2 \big/ c^2}$$

, which we can write as *m* +*dm*, so *dm* is the tiny mass increase we know must occur. It's easy to calculate *dm*. For small *v*, we can make the approximations

$$\sqrt{1 - v^2 \big/ c^2} \approx 1 - \frac{1}{2}\frac{v^2}{c^2}$$

and

$$\frac{1}{1 - \frac{1}{2}\frac{v^2}{c^2}} \approx 1 + \frac{1}{2}\frac{v^2}{c^2}$$

This means the total mass at speed *v* is *m*(1 + 1⁄2*v*∑/*c*∑), and writing this as *m* +*dm*, we see the mass increase *dm* equals 1⁄2 *mv*∑/*c*∑. This means that again, the mass increase *dm* is related to the kinetic energy *E* by $E = (dm)c^2$.

Kinetic Energy and Mass for Particles of Arbitrary Speed

We have shown in the two sections above that when a force does work to increase the kinetic energy of a particle it also causes the mass of the particle to increase by an amount equal to the increase in energy divided by c^2. In fact this result is exactly true over the whole range of speed from zero to arbitrarily close to the speed of light, as we shall now demonstrate.
For a particle of mass *m* accelerating along a straight line under a constant force *F*,

work done = force distance

so

Now

,

so

$$K.E. = \int \frac{d}{dt}(mv)ds = \int \frac{ds}{dt}d(mv) = \int v d(mv)$$

$$d(mv) = vdm + mdv = (v + m\frac{dv}{dm})dm$$

We find *dv*/*dm*:

$$m = \frac{m_0}{\sqrt{1 - v^2 / c^2}}, \frac{dm}{dv} = \frac{m_0 v / c^2}{(1 - v^2 / c^2)^{3/2}}$$

That is,

$$\frac{dm}{dv} = \frac{v / c^2}{1 - v^2 / c^2} \cdot \frac{m_0}{\sqrt{1 - v^2 / c^2}} = \frac{v}{c^2 - v^2} m$$

so

$$\frac{dv}{dm} = \frac{c^2 - v^2}{mv}$$

and

$$d(mv) = (v + \frac{c^2 - v^2}{\quad})dm$$

Therefore

So we see that in the general case the work done on the body, by definition its kinetic energy, is just equal to its mass increase multiplied by c^2.

To understand why this isn't noticed in everyday life, try an example, such as a jet airplane weighing 100 tons moving at 2,000mph. 100 tons is 100,000 kilograms, 2,000mph is about 1,000 meters per second. That's a kinetic energy 1⁄2Mv∑ of √.10^{11}joules, but the corresponding mass change of the airplane down by the factor c∑, 9.10^{16}, giving an actual mass increase of about half a milligram, not too easy to detect!

A Warning about Notation: m and m_0

We use m_0 to denote the "rest mass" of an object, and m to denote its relativistic mass,

$$m = \frac{m_0}{\sqrt{1 - v^2 / c^2}}.$$

In this notation, we follow French and Feynman. *Krane and Tipler, in contrast, use m for the rest mass*. Using m as we do gives neater formulas for momentum and energy, but is not without its dangers. One must remember that m is *not* a constant, but a function of speed. Also, *one must remember* that the relativistic kinetic energy is $(m-m_0)c^2$, and *not* equal to $\frac{1}{2}mv^2$, even with the relativistic mass!

Example: take $v^2/c^2 = 0.99$, find the kinetic energy, and compare it with $\frac{1}{2}mv^2$ (using the relativistic mass).

Physics 252 Home Page
Link to Next Lecture.

Copyright ©1997 Michael Fowler